Desensitized Optimal Trajectories

Hans Seywald
Renji Kumar

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Analytical Mechanics Associates, Inc.
303 Butler Farm Rd., Suite 104 A
Hampton, VA 23666
web: www.ama-inc.com
Phone: (757) 865-0944
email: seywald@ama-inc.com

Abstract

An optimal control formulation is introduced that enables the design of trajectories that are insensitive to state perturbations encountered at any time along the trajectory. Perturbations in fixed constants can be treated with this approach through a simple transformation that raises the status of the relevant parameters to that of states. Thus, the concepts introduced in this paper include a method to achieve robustness with respect to parameter uncertainties in the solution to nonlinear optimal control problems. As a numerical example, a Zermelo-type boat path optimization problem is treated, with uncertainties in the strength of the water current. Numerical solutions are generated with both, variational and direct optimization approaches.

1. Introduction

Guidance schemes are typically designed in two stages. In step one, a reference trajectory is generated that satisfies all the physical constraints and optimizes a user-chosen performance index. In step two, the equations of motion are linearized about the reference solution, and a linear control system is designed that drives the difference between actual states and reference states to zero. The main objectives in step two are to achieve stability, good command following, disturbance attenuation, and robustness with respect to unmodeled plant dynamics. Step 1 is called the trajectory optimization step [1, 2, 3] and step 2 is called the control loop design step [4, 5, 6].

Usually, the two design steps are performed completely independently. In particular, no sensitivity issues are considered during the design of the reference trajectory. Typically, the difficult task of making the real plant follow the prescribed reference solution is left completely to the linear control system.

In practice it may be crucial that certain properties of the resulting trajectory under feedback are insensitive with respect to perturbations in the state vector at any time along the trajectory. Designing the reference trajectory without any sensitivity considerations in mind may lead to a solution that requires the controller to operate dangerously close to its saturation point, or, controlling the system with the available control power may be impossible altogether. For example, for a rocket ascent problem we may wish to design the trajectory such that certain user-specified flight conditions near staging time are insensitive with respect to state perturbations induced by atmospheric wind gusts. For interplanetary fly-by and extraterrestrial landing missions, it may be desirable to achieve insensitivity of the final target point with respect to changes in environmental conditions such as solar pressure, gravitational fields, atmospheric pressure, etc.

The present paper introduces an optimal control problem formulation that enables the user to incorporate sensitivity issues of the type described above into the design of the optimal reference solution. It is well-known that the Lagrange multipliers in optimal control carry sensitivity information (see [1, 7]) and a first guess may be that the desired problem formulation is achieved by incorporating these Lagrange multipliers into the cost index. However, it is clear that Lagrange multipliers associated with the original problem (the problem without sensitivity penalties) do not correctly represent the sensitivities of the modified problem, and the Lagrange multipliers of the modofied problem (the problem with sensitivity penalties) can be generated only after the problem itself has been formulated. [4].

[§] This is a typical “chicken-and-egg” problem.

The approach introduced in this paper leads to an optimal control problem that involves controller feedback gains. Strictly speaking, this formalism allows an elaborate controller design in parallel to the design of the optimal reference trajectory. For practical applications, the effort and complexity involved in using this parallel approach to design a high-quality controller is probably not justified. It should be pointed out, however, that the introduction of feedback gains into the trajectory optimization process is dictated by the requirement to define control actions not only along the optimal reference solution, but within an open tube around the reference solution in order to lend any meaning to the sensitivity of the solution to state perturbations.

The paper is organized as follows: In Section 2 we define a standard optimal control problem and in Section 3 we review the formalism through which the sensitivity of a solution with respect to state perturbations can be captured. In section 4 this formalism is used to augment the original problem formulation of Section 2 through penalties on the sensitivity of the solution. In Sections 5 and 6, some generalizations and an alternative derivation of the new problem formulation of Section 4 are discussed. A numerical example is presented in Section 7.

2. Problem Formulation

Assume we are given a standard optimal control problem. In its simplest form, without state constraints, control constraints, or interior-point constraints, the problem can be stated as follows:

$$\min_{u\in(PWC[t_0,t_f])^m,t_f\in\mathbf{R}} \phi(x(t_f),t_f) \tag{1}$$

subject to

$$\dot{x} = f(x,u,t), \tag{2}$$

$$x(t_0) = x_0, \tag{3}$$

$$\psi(x(t_f),t_f) = 0, \tag{4}$$

with t_0 and x_0 fixed. Here, $(PWC[t_0,t_f])^m$ denotes the set of all piecewise continuous m-vector functions of time on the interval $[t_0,t_f]$, x denotes the state vector, and u denotes the control vector.

Once the optimal solution x^*, u^* of problem (1)–(4) is determined a control system has to be designed that tracks the reference solution in presence of unmodeled dynamics, external disturbances, and model uncertainties. Typically, the main objectives of this design are good command tracking and disturbance attenuation.

In Section 4, we will introduce a modification of the original problem formulation (1)–(4) such that the resulting optimal reference trajectory becomes well suited for the design of a feedback controller. The general approach to achieve this goal is to penalize the original cost function (1) with the sensitivity of user-specified quantities with respect to state perturbations encountered along the reference trajectory.

The significance of this concept becomes more clear by noting that parameter uncertainties, too, can be cast in terms of state perturbations. For any parameter p with the nominal value p_0 this is achieved by augmenting the representation (1–4) with the trivial initial value problem

$$\begin{array}{rcl} \dot{p} & = & 0, \\ p(t_0) & = & p_0. \end{array} \tag{5}$$

Before presenting the new optimal control formulation the following section summarizes a few mathematical prerequisites.

3. Review of Sensitivity Analysis

Let $x(t)\in\mathbf{R}$ obey the state equation

$$\dot{x} = g(x(t),t), \tag{6}$$

and let $X(t|t_0, x_0)$ denote the solution $x(t)$ of (6) subject to the initial condition

$$x(t_0) = x_0. \tag{7}$$

Furthermore, let $S(t|t_0, x_0)$ be the $n \times n$ matrix defined by

$$\frac{\partial\, S(t|t_0, x_0)}{\partial t} = \left.\frac{\partial\, g(x,t)}{\partial x}\right|_{x=X(t|t_0,x_0)} \cdot S(t|t_0, x_0) \tag{8}$$

$$S(t_0|t_0, x_0) = I. \tag{9}$$

The important property of the matrix $S(t|t_0, x_0)$ is that it represents the sensitivity of the state $X(t|t_0, x_0)$ with respect to perturbations in the initial state x_0 at time t_0 . Similarly, for any $t_1 \in [t_0, t_f]$ the inverse of $S(t|t_1, x_1)$, where $x_1 = X(t_1|t_0, x_0)$, represents the sensitivity of the solution X at time t_1 with respect to changes in the 'current state' $x(t)$ at time t. Noting that the Lagrange multiplier vector $\lambda(t)$ in optimal control represents the sensitivity of the terminal cost index $\phi(x(t_f), t_f)$ with respect to perturbations in $x(t)$ [1, 7], it is no surprise that the defining differential equation for $S(t|\cdot,\cdot)^{-1}$ is identical in structure to the costate equations associated with the system dynamics (2).

Below, the properties of S are summarized without proof :

1.

$$S(t|t_0, x_0) = \frac{\partial}{\partial x_0} X(t|t_0, x_0). \tag{10}$$

2. S is nonsingular for all t with

$$S(t|t_0, x_0)^{-1} = S(t_0|t, X(t|, t_0, x_0)). \tag{11}$$

3. For all t and $t_1 \in [t_0, t_f]$, $S(t|t_0, x_0)$ satisfies

$$S(t|t_0, x_0) = S(t|t_1, X(t_1|t_0, x_0)) \cdot S(t_1|t_0, x_0). \tag{12}$$

4. For all $t_1 \in [t_0, t_f]$ and $x_1 = X(t_1|t_0, x_0)$, $S(t|t_1, x_1)^{-1}$ satisfies the following initial value problem:

$$\frac{\partial\left(S(t|t_1, x_1)^{-1}\right)}{\partial t} = -S(t|t_1, x_1)^{-1} \cdot \left.\frac{\partial g(x,t)}{\partial x}\right|_{x=X(t|t_0,x_0)}, \tag{13}$$

$$S(t_1|t_1, x_1)^{-1} = I. \tag{14}$$

5. For all $t_1 \in [t_0, t_f]$ and $x_1 = X(t_1|t_0, x_0)$

$$S(t|t_1, x_1)^{-1} = \left.\frac{\partial X(t_1|t, x)}{\partial x}\right|_{x=X(t|t_0,x_0)}. \tag{15}$$

6. Let $h(x,t)$ be an arbitrary, smooth, scalar function of x and t. Furthermore, let $t_1 \in [t_0, t_f]$, $x_1 = X(t_1|t_0, x_0)$. Then the quantity $\Lambda(t, t_1)$ defined by

$$\Lambda(t, t_1) = \left.\frac{\partial h(x,t)}{\partial x}\right|_{\substack{x = x_1 \\ t = t_1}} S(t|t_1, x_1)^{-1} \tag{16}$$

satisfies

$$\frac{\partial}{\partial t}\Lambda(t, t_1)^T = -\Lambda(t, t_1)^T \left.\frac{\partial g(x,t)}{\partial x}\right|_{x=X(t|t_0,x_0)}, \tag{17}$$

$$\Lambda(t_1, t_1)^T = \left.\frac{\partial h(x,t)}{\partial x}\right|_{\substack{x = x_1 \\ t = t_1}}, \tag{18}$$

and represents the sensitivity of $h(x_1, t_1)$ with respect to perturbations in x at time t.

Clearly, statement 6 is the most important one for our purposes. If we are able to augment the state vector of the original optimal control problem (1)–(4) through an appropriate sensitivity matrix S, then the sensitivity of any smooth quantity $h(x(t_1), t_1)$, $t_1 \in [t_0, t_f]$ fixed, with respect to changes in the current state vector at time t is available through $\Lambda(t, t_1)$. This is clear from (15)–(16) as

$$\begin{aligned}\Lambda(t,t_1) &\overset{(16)}{=} \left.\frac{\partial h(x,t)}{\partial x}\right|_{\substack{x=x_1\\t=t_1}} S(t|t_1,x_1)^{-1}\\ &\overset{(15)}{=} \left.\frac{\partial h(x,t)}{\partial x}\right|_{\substack{x=x_1\\t=t_1}} \cdot \frac{\partial X(t_1|t,x(t))}{\partial x(t)}\\ &= \frac{\partial}{\partial x(t)} h(x(t_1),t_1). \end{aligned} \tag{19}$$

It is important to note that any sensitivity analysis of the nature described above requires calculation not only of the optimal reference trajectory but also (in some form) of a whole field of trajectories, at least in some neighborhood of the reference trajectory. Clearly, the concept of “sensitivity of the final cost with respect to state perturbations at time t” breaks down if it is not defined how to continue the trajectory after the perturbations have been applied. Through a linearity assumption on the control changes with respect to perturbations in the current state this leap from a single trajectory to a whole field of trajectories can be achieved by introducing constant or time-varying gain matrices to describe the control changes in the neighborhood of the reference trajectory. These gain matrices can be either prescribed a priori, or can be determined optimally in parallel to the calculation of the optimal reference trajectory. An optimal control problem formulation based on the concepts introduced above is presented in the next section.

4. Optimal Control Problem With Sensitivity Penalties

As a starting point we consider the standard optimal control problem (1)–(4) of Section 2. Instead of minimizing only the original cost index $\phi(x(t_f), t_f)$, we are now interested also in reducing the sensitivity of a user-specified function $h(x(t_f), t_f)$ of final states and final time with respect to state perturbations encountered along the trajectory. Applying the results of the previous section we can state the following optimal control problem:

Find the control functions of time $u \in (PWC(t_0, t_f])^m$, the gain functions of time $K \in (PWC(t_0, t_f])^{m \times n}$, and the final time t_f such that the cost function

$$J = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} ||h_x(x_f, t_f) \cdot S_f \cdot S(t)^{-1}||^2_{Q(t)}\, dt. \tag{20}$$

is minimized subject to the constraints

$$\dot{x} = f(x, u, t) \tag{21}$$

$$x(t_0) = x_0 \tag{22}$$

$$\psi(x(t_f), t_f) = 0 \tag{23}$$

$$\dot{S} = \frac{\partial f}{\partial x} \cdot S + \frac{\partial f}{\partial u} \cdot K \cdot S \tag{24}$$

$$S(t_0) = I_{n\times n}, \tag{25}$$

$$K_{min} \le K \le K_{max}, \tag{26}$$

where x is n-dimensional, u is m-dimensional, ψ is k-dimensional, and S is an $n \times n$ square matrix. x, u, t, t_0, t_f, and x_f denote the state vector, the control vector, time, initial time, final time, and the final states, respectively, and $S_f = S(t_f)$. Equations (21), (22), (23) represent the state equations, initial conditions, and final conditions of the original optimal control problem (1)–(4), respectively. The matrix inequality constraint (26) is understood component wise. $Q(t)$ is a user-chosen positive semi-definite weighting matrix function of time, and $||\cdot||_{Q(t)}$ denotes the Q-norm defined by $||\cdot||_{Q(t)} = \sqrt{(\cdot)^T Q(t)\,(\cdot)}$. Through the feedback law

$$u(t) = u^*(t) + K^*(t)\,(x(t) - x^*(t))\,, \tag{27}$$

where superscript * denotes quantities associated with the optimal solution to problem (20)–(27), the $m \times n$ gain matrix $K(t)$ defines the control actions in an open tube around the optimal reference solution¶. In the same way that equations (8, 9) define a sensitivity matrix for the equations of motion (6), equations (24, 25) define a sensitivity matrix for the underlying state dynamics

$$g(x(t), t) = f\left(x(t), u^*(t) + K^*(t)\,(x(t) - x^*(t))\,, t\right). \tag{28}$$

Note also that, in the present section, we have introduced the notation $S(t) := S(t|t_0, x_0)$, i.e. it is tacitly understood that the second and third component in $S(t|\cdot,\cdot)$ are always t_0 x_0, respectively.

To understand the physical significance of the inegral term in the cost function (20) we use equations (11), (12) to express the quantity $S(t|t_f, x_f)^{-1}$, where $x_f = X(t_f|t_0, x_0)$, in the following form:

$$\begin{aligned} S(t|t_f, x_f)^{-1} &\overset{(11)}{=} S(t_f|t, X(t|t_0, x_0)) \\ &\overset{(12)}{=} S(t_f|t_0, x_0) \cdot S(t_0|t, X(t|t_0, x_0)) \\ &\overset{(11)}{=} S(t_f|t_0, x_0) \cdot S(t|t_0, x_0)^{-1} \\ &= S(t_f) \cdot S(t)^{-1}. \end{aligned} \tag{29}$$

Hence, following (16), for any smooth function $h(x(t_f), t_f)$ the sensitivity $\Lambda(t, t_f)$ of this function with respect to changes in the state x at time t can be represented in the form

$$\Lambda(t, t_f) = h_x(x(t_f), t_f) \cdot S(t_f) \cdot S(t)^{-1}. \tag{30}$$

While the original problem (1)–(4) is given in standard Meyer form the new problem formulation (20)–(27) is given in Lagrange form. However, as the integrand in the cost index (20) does not only depend on the controls, states, and time, but also on the final states x_f, S_f, and the final time t_f, the formulation (20)–(27) is not standard. Following well-known procedures, a standard Lagrange formulation can be achieved by introducing additional states that represent the values of x_f, S_f, and T_f. Explicitly, this can be achieved by defining the additional states $x_f(t) \in \mathbf{R}^n$, $S_f(t) \in \mathbf{R}^{n,n}$, $T_f(t) \in \mathbf{R}^1$ through

$$\dot{x_f} = 0, \quad \dot{S_f} = 0, \quad \dot{T_f} = 0, \tag{31}$$

subject to the boundary conditions

$$x_f(t_f) = x(t_f), \quad S_f(t_f) = S(t_f), \quad T_f(t_f) = t_f, \tag{32}$$

respectively. To transform the resulting standard Lagrange problem to a standard Meyer problem one can additionally introduce a cost state y through

$$\dot{y} = \frac{\partial\phi}{\partial x} \cdot f(x, u, t) + \frac{\partial\phi}{\partial t} + ||h_x(x_f, T_f) \cdot S_f(t) \cdot S(t)^{-1}||^2_{Q(t)}, \tag{33}$$

$$y(t_0) = 0, \tag{34}$$

and replace the performance index (20) through

$$J = y(t_f). \tag{35}$$

¶Note: the superscript * on K in equation (27) indicates that K is optimized in parallel to the optimization of u. As stated earlier, problem (20)–(27) can be simplified considerably by prescribing $K(t)$ as a fixed function of time.

5. Some Generalizations

Extension 1: The problem formulation (20)–(27) enables the design of an optimal trajectory that reduces the sensitivity of a user-specified quantity $h(x(t_f), t_f)$ with respect to perturbations encountered during the course of the trajectory. In (20) we added the respective sensitivities of h to the original cost function $\phi(x(t_f), t_f)$ in a weighted integral square sense. In (33)–(35) it was seen how the so-obtained cost index can be transformed into Meyer form. Clearly, the formalism of the previous section remains correct if we relax the special form of the cost function (20) and if we allow the right-hand side of the cost state y in (33) to be an arbitrary function of x, u, t, S, x_f, S_f, and t_f.

Extension 2: It is clear that the construction of an appropriate cost index $J[x, u]$ as obtained in (20) can be easily generalized to the case where a user wishes to involve any finite number of functions $h_i(x(t_f), t_f)$, $i = 1, .., L$, into the sensitivity considerations. This would only complicate the cost index (20), or equivalently, the right-hand side of the cost state (33), but it would have no effect on the general structure of (20–27). In particular, the sensitivity matrix S needs to be carried along only once, irrespective the number L of expressions $h_i(x(t_f), t_f)$ involved in the problem.

Extension 3: To involve the sensitivity of a function of states and time, evaluated at a time different from the final time, say, t_1, it is necessary to perform some minor modifications. In analogy to equation (30) the sensitivity of a quantity $h(x, t)$ evaluated at time t_1 with respect to perturbations applied in the states at time t prior to t_1 are represented by

$$\Lambda(t, t_1) = h_x(x(t_1), t_1) \cdot S(t_1) \cdot S(t)^{-1}. \tag{36}$$

The sensitivity of $h(x(t_1), t_1)$ with respect to perturbations applied in the states at time t after t_1 is obviously zero.

6. An Alternative Derivation

In Section 4 the gain matrix $K(t)$ was introduced somewhat in an ad hoc fashion. Without further proof it was stated that any sensitivity analysis would require defining the control actions not only along the optimal reference solution but also in some neighborhood of this solution. K was then introduced as the tool to define these control actions through equation (27). In the present section it is attempted to shed more light on this issue by approaching the optimal control problem from a very general point of view.

In the spirit of the general problem formulation outlined in Section 2, we can state that we wish to determine the feedback control law $u = U(x, t)$ such that the cost functional

$$J[u] = Y(t_f|t_0, x_0) \tag{37}$$

is minimized for all fixed t_0, x_0, subject to the constraints

$$\frac{\partial}{\partial t} X(t|t_0, x_0) = f(X(t|t_0, x_0), U(X(t|t_0, x_0), t), t) \tag{38}$$

$$X(t_0|t_0, x_0) = x_0 \tag{39}$$

$$\psi(X(t_f|t_0, x_0), t_f) = 0 \tag{40}$$

$$S(t|t_0, x_0) = \frac{\partial}{\partial x_0} X(t|t_0, x_0) \tag{41}$$

$$\frac{\partial}{\partial t} Y(t|t_0, x_0) = F(X(t|t_0, x_0), U(X(t|t_0, x_0), t), S(t_f|t, X(t|t_0, x_0)), t) \tag{42}$$

$$Y(t_0|t_0, x_0) = 0. \tag{43}$$

From Section 3 we find that the sensitivity matrix S defined in (41) satisfies

$$S(t_0|t_0, x_0) = I, \tag{44}$$

$$\frac{\partial S(t|t_0, x_0)}{\partial t} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\cdot\frac{\partial U}{\partial x}\right)\cdot S(t|t_0, x_0). \tag{45}$$

Above, the partials $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial u}$, $\frac{\partial U}{\partial x}$ are evaluated at $x = X(t|t_0, x_0)$ and $u = U(X(t|t_0, x_0), t)$, i.e.

$$\frac{\partial f}{\partial x} = \left.\frac{\partial f(x, u, t)}{\partial x}\right|_{\substack{x = X(t|t_0, x_0)\\ u = U(X(t|t_0, x_0), t)}}, \tag{46}$$

$$\frac{\partial f}{\partial u} = \left.\frac{\partial f(x, u, t)}{\partial u}\right|_{\substack{x = X(t|t_0, x_0)\\ u = U(X(t|t_0, x_0), t)}}, \tag{47}$$

$$\frac{\partial U}{\partial x} = \left.\frac{\partial U(x, t)}{\partial x}\right|_{x = X(t|t_0, x_0)}. \tag{48}$$

For some fixed initial conditions $x(t_0) = x_0$ let superscript * now denote the optimal solution, i.e.

$$x^*(t) = X(t|t_0, x_0), \tag{49}$$

$$u^*(t) = U(x, t)|_{x = x^*(t)}. \tag{50}$$

Along this optimal trajectory, the expression

$$\frac{\partial U}{\partial x} = \left.\frac{\partial U(x, t)}{\partial x}\right|_{x = x^*(t)} \tag{51}$$

becomes simply a function of time. Loosely speaking, $\frac{\partial U}{\partial x}$ above can be identified as the gain function of time, $K(t)$ introduced in Section 3. It should be noted however that the boundary conditions (40) are much stronger than those in (23). Condition (40) requires that the reference solution and all perturbed solutions perfectly satisfy the boundary conditions $\psi(x(t_f), t_f) = 0$. In contrast, (23) enforces this condition only for the reference solution. In general, condition (40) leads to the unboundedness of $\frac{\partial U}{\partial x}$ near the final time t_f. Clearly, to achieve satisfaction of (40) in the formulation (20)–(26) would require an additional boundary condition on $S(t_f)$, e.g. for fixed final time problems

$$\frac{\partial \psi(x(t_f), t_f)}{\partial x(t_f)} \cdot S(t_f) = 0. \tag{52}$$

For nontrivial boundary conditions (23) this would require $S(t_f)$ to be singular, which is impossible as long as the right-hand side of (24) is bounded. Thus, problem (20)–(26) represents a weaker but also more practical formulation than problem (37)–(43).

7. Numerical Example

Given is a river flowing in x_1-direction. The shore line lies along $x_1 = 0$, and the velocity of the water current increases linearly with the distance from the shore, i.e. $v_{current} = p \cdot x_2$ for some constant p. For the numerical calculations we use

$$p = 10. \tag{53}$$

At initial time, the boat starts at point A with the coordinates $x_1 = x_2 = 0$, and we wish to place the target point B (a "bus stop"), which has to be reached after a fixed time of, say, one time unit, as far as possible downstream along the shore line (see Figure 1). In Meyer form this optimal control problem can be stated as follows:

$$\min_{u, K \in PWC[0, t_f]} -x_1(t_f) \tag{54}$$

$$\begin{aligned} \dot{x}_1 &= \cos u + p \cdot x_2 \\ \dot{x}_2 &= \sin u \end{aligned} \tag{55}$$

$$\begin{aligned} &x_1(0) = 0, \quad x_2(0) = 0, \\ &x_2(t_f) = 0, \quad t_f = 1. \end{aligned} \tag{56}$$

The optimal solution to this problem is presented in Figure 2.

In practice, the water current represented by the parameter p may not be known precisely, or it may even change during the boat ride. Any change in p is likely to affect the precision with which the target point B is hit. We want to design the trajectory such that the target point is still located reasonably far downstream, but also such that any changes in the parameter p encountered along the trajectory have only a reasonably small effect on the precision with which the nominal target point is hit.

To enable the consideration of sensitivities with respect to changes in p through the formalism introduced in this paper we need to raise the status of the parameter p to that of a state. This is achieved by introducing the additional state x_3, representing p, through $\dot{x}_3 = 0$, $x_3(0) = 10$. Along the trajectory of the boat we wish to penalize the sensitivities of the final states $x_1(t_f)$ and $x_2(t_f)$ with respect to changes in the state $x_3(t)$ in an integral square sense. From (30) and some simple calculations involving $h_1(x,t) = x_1$ and $h_2(x,t) = x_2$ it is clear that these sensitivities are represented by the $(1,3)$- and $(2,3)$-elements of the 3×3-matrix function $S(t_f) \cdot S(t)^{-1}$. Following the general approach outlined in Section 4, the formal optimal control problem replacing problem (54 - 56) can then be stated as follows:

$$\min_{u,K \in (PWC[0,t_f])^3} y(t_f) \tag{57}$$

subject to the constraints

$$\dot{y} - cosu - x_2 x_3 + \alpha \left(\left(S_f S^{-1} \right)^2_{1,3} + \left(S_f S^{-1} \right)^2_{2,3} \right), \tag{58}$$

$$y(0) = 0, \tag{59}$$

$$\begin{aligned} \dot{x}_1 &= \cos u + x_3 \cdot x_2, \\ \dot{x}_2 &= \sin u, \\ \dot{x}_3 &= 0, \end{aligned} \tag{60}$$

$$x_1(0) = 0, \;\; x_2(0) = 0, \;\; x_3(0) = 10, \tag{61}$$

$$x_2(t_f) = 0, \;\; t_f = 1, \tag{62}$$

$$\dot{S} = \begin{bmatrix} -K_1 \sin u & -K_2 \sin u + c & x_2 \\ K_1 \cos u & K_2 \cos u & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot S, \tag{63}$$

$$S(0) = I_{3\times 3}, \tag{64}$$

$$\dot{S}_f = 0_{3\times 3}, \tag{65}$$

$$S_f(t_f) = S(t_f). \tag{66}$$

Equations (60)–(62) represent the state dynamics and boundary conditions of the original optimal control problem (53)–(56), equations (63), (64) define the sensitivity matrix S, and conditions (58), (59) define the cost state y. The additional conditions (65, 66) are used to raise the status of the components of the matrix S_f to that of states, so that the resulting problem formulation is given in standard Meyer form. Note that S_f appears in the right-hand side of (58). In general, also the final state x_f could appear on the right-hand side of the $\dot{y}$-equation (see (33)). If this is the case in a particular numerical example then further augmentation of the state vector is required in an obvious way to preserve the Meyer form.

For numerical calculations, the gain functions K_1 and K_2 are bounded between ± 1, i.e.

$$-1 \leq K_i \leq +1, \quad i = 1, 2. \tag{67}$$

and the penalty parameter α is varied between 0 and 10,000. Figure 3 shows the associated optimal solutions in the x_1, x_2-plane. It is observed that for $\alpha = 0$ the optimal solution to problem (57)–(67) is identical to the optimal solution to problem (54)–(56) in the states x_1 and x_2. The optimal gain functions K_1 and K_2 are always identically -1.

Figures 4 and 5 show the individual sensitivities $\left(S(t_f)S(t)^{-1}\right)_{1,3}$ and $\left(S(t_f)S(t)^{-1}\right)_{2,3}$ as functions of time, respectively, and Figure 6 shows the cost term $\left(S(t_f)S(t)^{-1}\right)_{1,3}^2 + \left(S(t_f)S(t)^{-1}\right)_{2,3}^2$ as a function of time. It is interesting to note that for $\alpha = 1000$, x_2 takes on negative values near the end of the trajectory, thus making use of the reverse water current to slow down the boat. This seems to be a somewhat non-intuitive result, and it motivated the authors do confirm the obtained results using a direct optimization approach. For a node density of 41 nodes (40 subintervals) the associated results obtained with a collocation-based technique are depicted in Figure 7 and it can be seen that the direct solution agrees well with the variational solution. In this context, it is important to note that the direct and indirect solutions, including the initial guesses, were developed completely independently from each other.

Summary

A method was introduced to include sensitivity considerations into the design of optimal trajectories. The approach presented here involves the design of the reference solution along with a family of trajectories in the neighborhood of this solution. In fact, one of the main claims in this paper is that sensitivity considerations are impossible without the design of trajectory fields.

In this paper only the most basic optimal control problem is treated. State and control constraints have not been considered. As a numerical example, a Zermelo-type boat path optimization problem was solved with penalties on the sensitivity of the reference solution with respect to changes in the water current.

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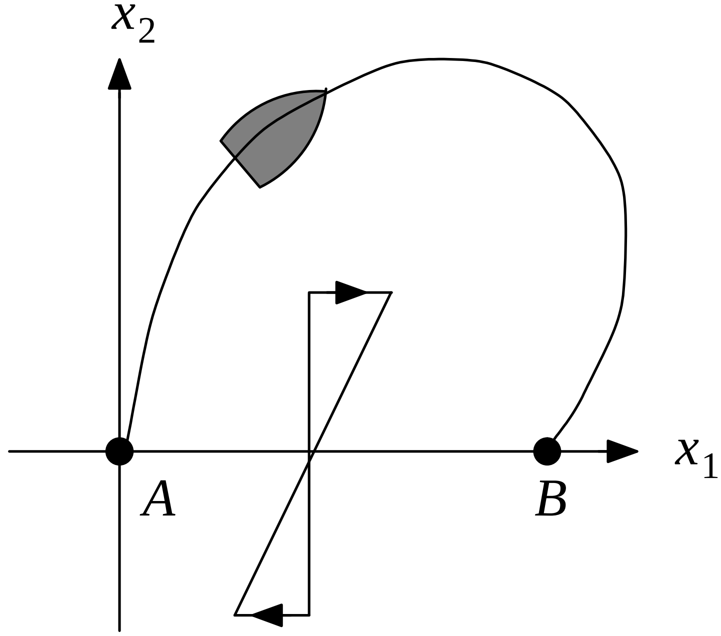

Figure 1: Physical problem description

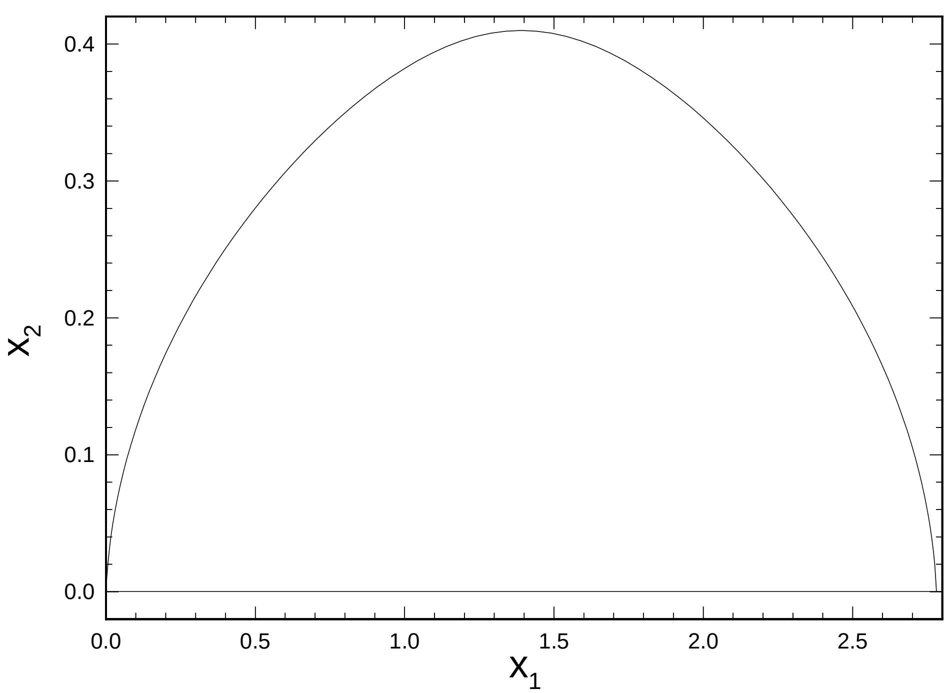

Figure 2: Optimal solution to original problem without sensitivity considerations

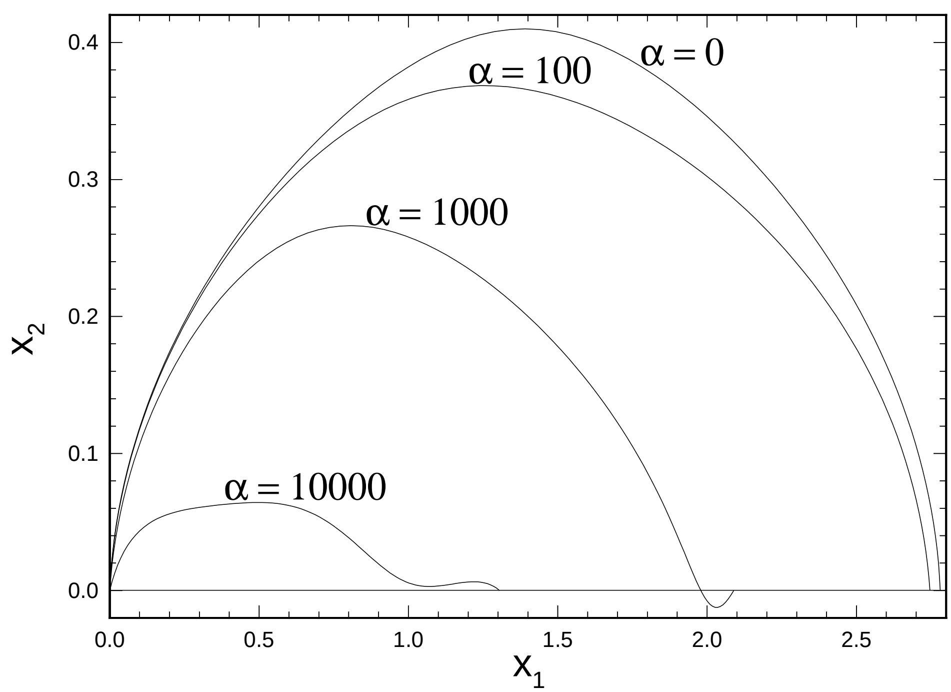

Figure 3: Optimal solutions obtained with various penalty factors on sensitivity

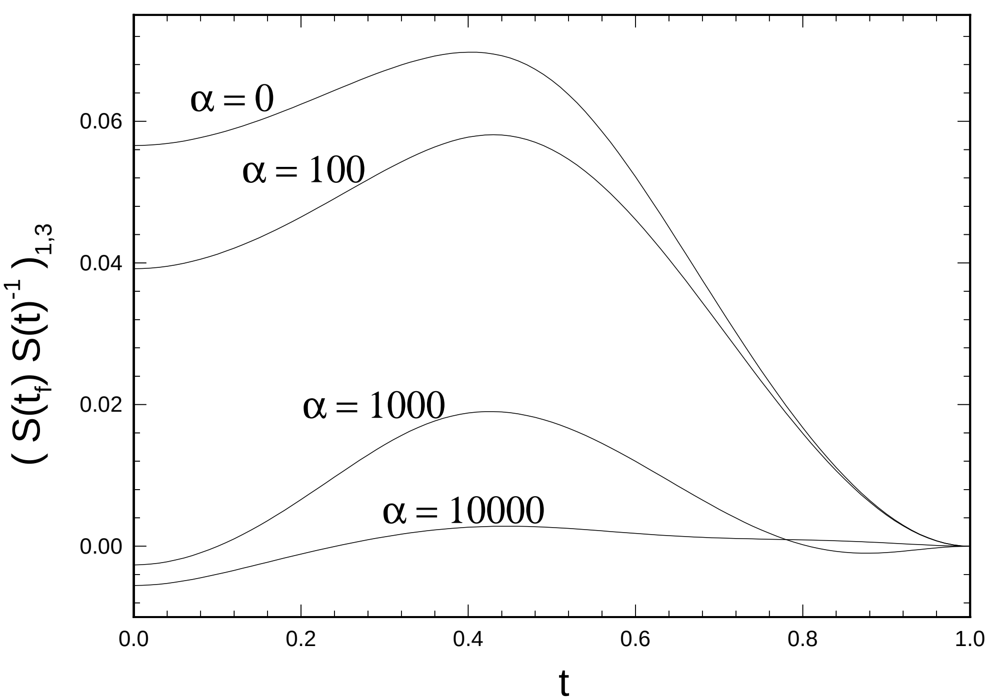

Figure 4: Sensitivity of state 1 with respect to perturbations in state 3

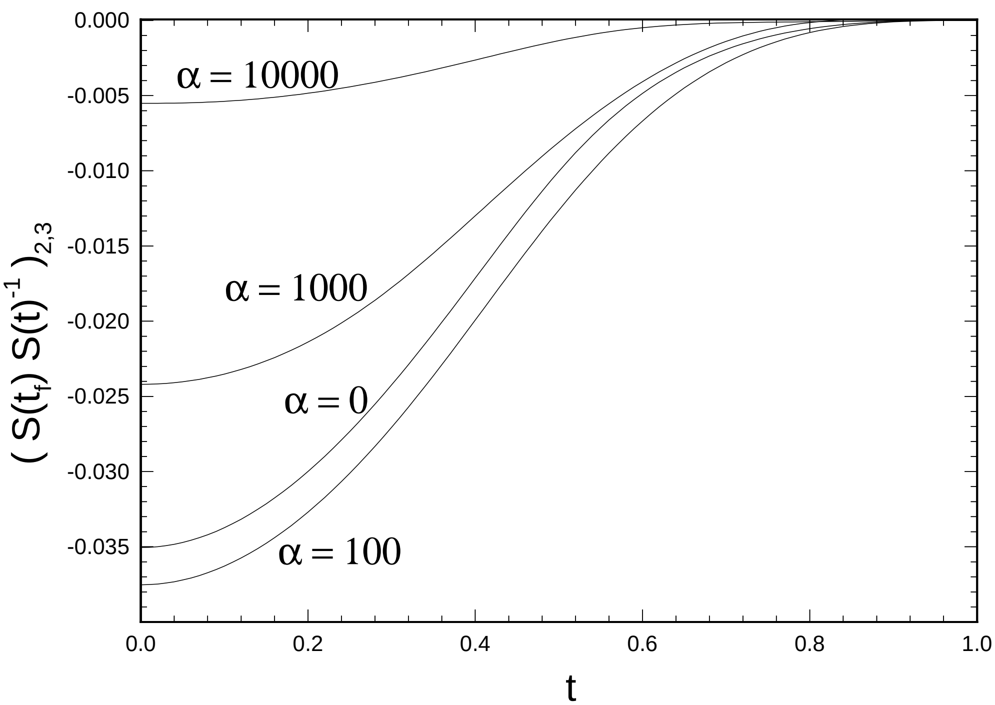

Figure 5: Sensitivity of state 2 with respect to perturbations in state 3

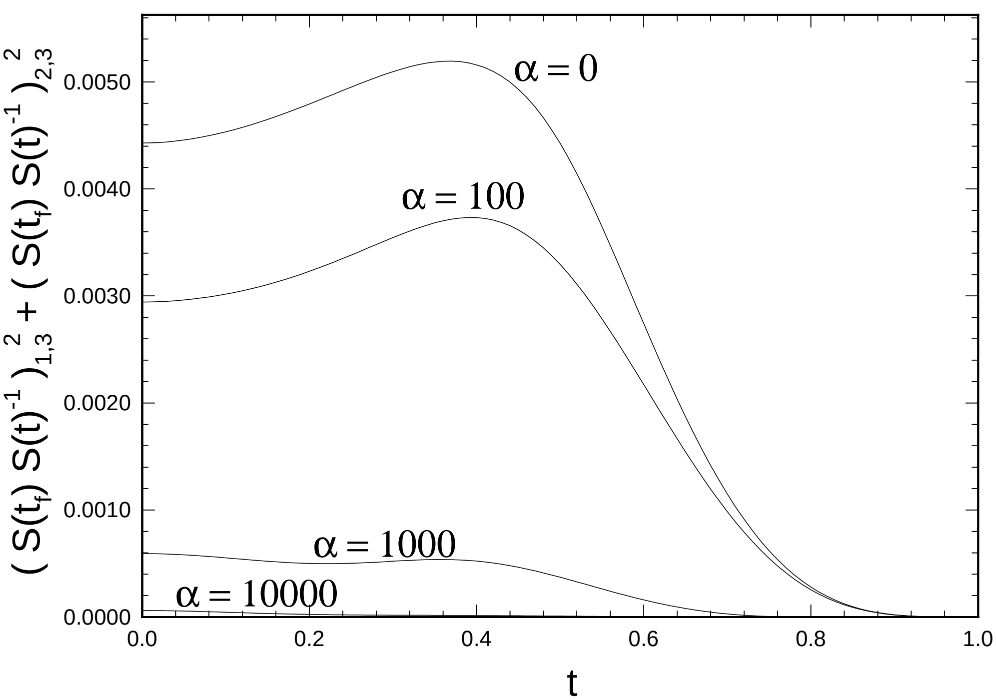

Figure 6: Total sensitivity penalty - without the penalty factor α

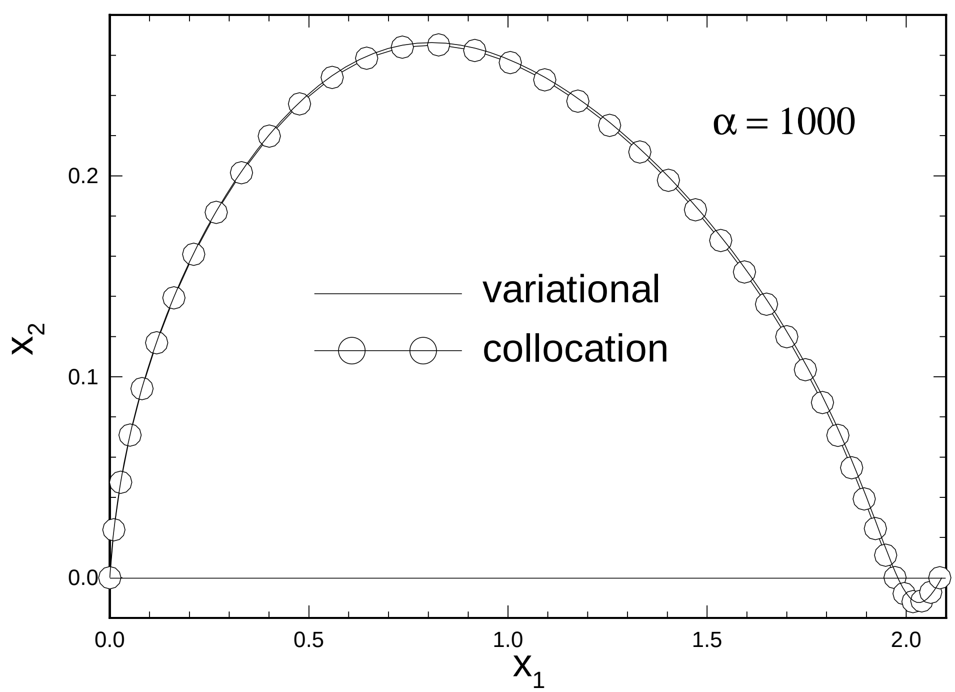

Figure 7: Comparison between variational solution and collocation solution